Math 5C Discussion Problems 2 Selected Solutions

Path Independence

1. Let C be the striaght-line path in $\mathbb{R}^2$ from the origin to $(3,1)$. Define $f(x,y) = xye^{xy}$.

   (a) Evaluate $\int_C \nabla f \cdot d\mathbf{r}$.

   Solution. $\int_C \nabla f \cdot d\mathbf{r} = f(3,1) - f(0,0) = 3e^3.$ □

   (b) Evaluate $\int_C ((1,0) + \nabla f) \cdot d\mathbf{r}$.

   Solution. $\int_C ((1,0) + \nabla f) \cdot d\mathbf{r} = \int_C \nabla(x) \cdot d\mathbf{r} + 3e^3 = 3 + 3e^3.$ □

   (c) Evaluate $\int_C ((y,0) + \nabla f) \cdot d\mathbf{r}$.

   Solution. $\int_C ((y,0) + \nabla f) \cdot d\mathbf{r} = \int_0^1 3t\,dt + 3e^3 = \frac{3}{2} + 3e^3.$ □

3. For each of the following vector fields, determine whether it is the gradient of a function.

   (a) $(4x^2 - 4y^2 + x, 7xy + \ln y)$ on $\mathbb{R}^2$

   Solution. No, because $\frac{\partial}{\partial y}(4x^2 - 4y^2 + x) \neq \frac{\partial}{\partial x}(7xy + \ln y)$. □

4. For each of the following, find the function f.

   (a) $f(0,0,0) = 0$ and $\nabla f = (x,y,z)$

   Solution. For any C from $(0,0,0)$ to (x,y,z), we have $f(x,y,z) = f(0,0,0) + \int_C \nabla f \cdot d\mathbf{r}$. Take $\mathbf{r}(t) = (xt, yt, zt)$, $0 \leq t \leq 1$ to get $f(x,y,z) = 0 + \int_0^1 (xt, yt, zt) \cdot (x,y,z)\,dt = (1/2)(x^2 + y^2 + z^2)$. □

6. Let C be a curve in $\mathbb{R}^2$ given by $\mathbf{r}(t) = (\cos^5 t, \sin^3 t, t^4)$, where $0 \leq t \leq \pi$. Evaluate $\int_C (yz, xz, xy) \cdot d\mathbf{r}$.

   Solution. Note that $(yz, zx, xy) = \nabla(xyz)$. The endpoints are $(1,0,0)$ and $(-1,0,\pi^4)$, so the integral is $(-1)(0)(\pi^4) - (1)(0)(0) = 0$. □

Green's Theorem

1. Let D be the unit disk centered at the origin in $\mathbb{R}^2$.

   (a) $\displaystyle\oint_{\partial D} dx + x\,dy = \pi.$

   (b) $\displaystyle\oint_{\partial D} \arctan(e^{\sin x})\,dx + y\,dy = 0.$

   (c) $\displaystyle\oint_{\partial D} (x^3 - y^3)\,dx + (x^3 + y^3)\,dy = \iint_D 3(x^2 + y^2)\,dA = \frac{3\pi}{2}.$

   (d) $\displaystyle\oint_{\partial D} \frac{\partial f}{\partial y}\,dx - \frac{\partial f}{\partial x}\,dy = 0$, given that $f_{xx} + f_{yy} = 0$.

2. Find the area of the following regions in $\mathbb{R}^2$ using Green's theorem.

   Hint. Use the formulas

$$\text{area}(R) = \oint_{\partial R} y\,dx = \oint_{\partial R} -x\,dy = \frac{1}{2}\oint_{\partial R} -x\,dy + y\,dx$$

□

4. Let f be a smooth function and D be a disk in $\mathbb{R}^2$ with outward unit normal $\mathbf{n}$. For points on ∂D, denote $\partial f/\partial n$ to mean the directional derivative of f in the direction of $\mathbf{n}$. Prove that

$$\oint_{\partial D} \frac{\partial f}{\partial n}\,ds = \iint_D \Delta f\,dA.$$

   Hint. Notice that $\partial f/\partial n = \nabla f \cdot \mathbf{n}$. Use the normal form of Green's theorem. □

5. Prove the identity $\displaystyle\oint_{\partial D} f\nabla f \cdot \mathbf{n}\,ds = \iint_D (f\Delta f + \|\nabla f\|^2)\,dA.$

   Hint. Remember that $\|u\|^2 = u \cdot u$. Use the normal form of Green's theorem and a product rule. □

Divergence Theorem

1. In each of the following situations, evaluate $\iint_{\partial R} \mathbf{F} \cdot d\mathbf{A}$. Assume ∂R is outward oriented.

   (a) Let R be the unit ball centered at the origin and $\mathbf{F} = (x, 2y, 3z)$.

   Solution. By the divergence theorem, $\iint_{\partial R} \mathbf{F} \cdot d\mathbf{A} = \iiint_R \nabla \cdot \mathbf{F}\, dV = \iiint_R 6\, dV = 8\pi$. □

2. Let S_1 be the disk $x^2 + y^2 \leq 1$, $z = 1$, oriented upward. Let S_2 be the cone $x^2 + y^2 = z^2$, $0 \leq z \leq 1$, oriented downward. Together, S_1 and S_2 enclose a region R. Define $\mathbf{F} = (x + e^y, y + \cos x, z)$.

   (a) Find the flux of $\mathbf{F}$ across S_1 directly.

   (b) Integrate $\nabla \cdot \mathbf{F}$ over R.

   (c) With no extra computation, find the flux of $\mathbf{F}$ across S_2. Do you see how this problem could be generalized?

   Solution. On S_1, the unit normal is $(0, 0, 1)$ and $\mathbf{F} = (x + e^y, y + \cos x, 1)$, so the flux is

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{A} = \iint_{S_1} (x + e^y, y + \cos x, 1) \cdot (0, 0, 1)\, d\sigma = \iint_{S_1} d\sigma = \pi.$$

   Since $\nabla \cdot \mathbf{F} = 2$, using the volume of a cone gives

$$\iiint_R \nabla \cdot \mathbf{F}\, dV = \iiint_R 2\, dV = 2(1/3)\pi(1)(1)^2 = \frac{2\pi}{3}.$$

   Finally, since $\partial R = S_1 \cup S_2$, the divergence theorem gives

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{A} = \iiint_R \nabla \cdot \mathbf{F}\, dV - \iint_{S_1} \mathbf{F} \cdot d\mathbf{A} = -\frac{\pi}{3}.$$ □

3. In each of the following, use the method of the previous problem to find the flux of $\mathbf{F}$ across S.

   (a) Let S be the hemisphere $x^2 + y^2 + z^2 = 9$, $z \geq 0$, outwardly oriented. Assume $\mathbf{F} = (x^2, 0, 2z)$.

   Answer. 36π □

   (b) Let S be the cone $z = 4 - \sqrt{x^2 + y^2}$, $z \geq 0$, oriented upward . Assume $\mathbf{F} = (xy, yz, xz)$.

   Answer. $64\pi/3$ □

4. *Hint.* For all of these, use the divergence theorem and product rules. □

5. Standard integration by parts is proved using the product rule. But there are many different product rules; each gives a different integration by parts.

   (b) Now let D be the unit ball centered at the origin. Evaluate

$$\iiint_D e^{-\sqrt{x^2+y^2+z^2}}\, \nabla \cdot \left(\frac{(x, y, z)}{(x^2 + y^2 + z^2)^{3/2}} \right) dV.$$

   You can ignore the singularities at the origin (this could be made rigorous).

   Answer. 4π □

Stokes' Theorem

1. In each of the following situations, evaluate $\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A}$.

   (a) Let S be the upper half ($z \geq 0$) of the sphere $x^2+y^2+z^2 = 1$, oriented upward, and $\mathbf{F} = (x, xz, ye^{\cos y})$.

   Solution. Note that ∂S is the unit circle (positively oriented) in $\mathbb{R}^2$ centered at the origin. On ∂S, $\mathbf{F} = (x, 0, ye^{\cos y})$ and $d\mathbf{r} = (dx, dy, 0)$, so $\mathbf{F} \cdot d\mathbf{r} = x\,dx$. Notice that $(x, 0, 0)$ is path-independent; thus Stokes' theorem gives

$$\iint_S \nabla \times \mathbf{F} \cdot d\mathbf{A} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \oint_{\partial S} x\,dx = 0.$$ □

2. Let C be the interesection of a (nonvertical) plane and the cylinder $x^2 + y^2 = 4$ in $\mathbb{R}^3$. Show that

$$\oint_C (2x - y)\,dx + (2y + x)\,dy = 8\pi.$$

   Hint. Note that $\nabla \times (2x - y, 2y + x, 0) = (0, 0, 2)$. Use Stokes theorem. □

4. *Hint.* Use Stokes theorem and product rules. Don't forget that the cross product is anti-commutative: $a \times b = -b \times a$. Also, the curl of a gradient is zero. □

5. Standard integration by parts is proved using the product rule. But there are many different product rules; each gives a different integration by parts.

   (a) Let S be a smooth oriented surface with boundary ∂S. Given a smooth vector field $\mathbf{G}$ and a smooth scalar function f, show that

$$\iint_S f(\nabla \times \mathbf{G}) \cdot d\mathbf{A} = -\iint_S (\nabla f \times \mathbf{G}) \cdot d\mathbf{A} + \oint_{\partial S} f\mathbf{G} \cdot d\mathbf{r}.$$

   (b) Now let S be the cone $z = \sqrt{x^2 + y^2}$, $0 \leq z \leq 1$, oriented downward. Define $\mathbf{G} = (-y, x, \arctan(xyz)e^{x^2})$ and evaluate

$$\iint_S z^2\,(\nabla \times \mathbf{G}) \cdot d\mathbf{A}$$

   (c) (Harder) Recall the identity $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$. Prove the vector equation

$$\iint_S \nabla f \times d\mathbf{A} = -\oint_{\partial S} f d\mathbf{r}.$$

   Hint, etc. Use the product rule for the first part. The answer to the second part is π (you can use the previous part or the divergence theorem—try both!) For the final part, note that vectors a and b are equal if and only if $a \cdot v = b \cdot v$ for all arbitrary vectors v. Take $\mathbf{G}$ in the first part to be an arbitrary constant vector field $\mathbf{c}$ and play around. □

Sequences and Series

1. Find the limit of the following sequences.

   (a) $a_n = \ln n/n \to 0$

   (b) $a_n = (1 - 2/n)^{3n} \to e^{-6}$

   (c) $a_n = \sqrt{n^2 + 3n} - n \to 3/2$

2. Given the sequence (a_n):

$$\sqrt{1}, \sqrt{1+\sqrt{1}}, \sqrt{1+\sqrt{1+\sqrt{1}}}, \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}, \ldots$$

   show that $a_{n+1}^2 - 1 = a_n$. Given that the limit of the sequence exists, use this formula to find it.

   Hint. Calling the limit L, we have $L^2 - 1 = L$. □

3. Evaluate the following series.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{2^{3n}}{3^{2n}} = 8$

   (b) $1 + \sin^2\theta + \sin^4\theta + \sin^6\theta + \cdots = \sec^2\theta$, where $0 < \theta < \pi/2$

   (c) $\displaystyle\frac{14}{15} + \frac{28}{75} + \frac{56}{375} + \frac{112}{1875} + \cdots = \frac{14}{9}$

4. Evaluate the following series. (*These all telescope*)

   (a) $\displaystyle\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots = 1$ (use partial fractions)

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n^2+n}} = 1$

   (c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n(n+2)}{(n+1)^2}\right) = -\ln 2$

   (d) $\displaystyle\sum_{n=0}^{\infty} \arctan(n+1) - \arctan(n) = \frac{\pi}{2}$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1} + (n+1)\sqrt{n}} = 1$

5. A difficult series to evaluate is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Use this fact to evaluate the following.

   (a) $\displaystyle\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{24}$

   (b) $\displaystyle 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$

Convergence of Series

1. Determine the convergence of the following series.

   (a) $\sum \frac{\sqrt{n}}{n^2}$ converges by p-series

   (b) $\sum \frac{\sqrt{n}}{1+n^2}$ converges by comparison to p-series

   (c) $\sum \left(1+\frac{1}{n}\right)^{-n}$ diverges since terms do not tend to 0

   (d) $\sum \left(1-\frac{1}{n}\right)^{n^2}$ converges by root test

   (e) $\sum \sin(1/n)$ diverges by limit comparison to harmonic series

   (f) $\sum n^2 e^{-n^3}$ converges by ratio or root or integral or ...

   (g) $\sum \frac{\arctan n}{n^2}$ converges by comparison to Euler's series

   (h) $\sum \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges by comparison to p-series or by telescoping

   (i) $\sum \frac{k^{-1/2}}{1+\sqrt{k}}$ diverges by comparison to harmonic series

   (j) $\sum \left[\left(\frac{n+1}{n}\right)^{n+1} - \left(\frac{n+1}{n}\right)\right]^{-n}$ converges by root test

2. Determine whether each series converges absolutely, conditionally, or not at all.

   (a) $\sum \frac{(-1)^n}{n \ln n}$ conditionally, not absolutely

   (b) $\sum \frac{(-4)^n}{n^2}$ diverges

   (c) $\sum \frac{\cos(n\pi)}{n}$ conditionally, not absolutely

3. Use the error bound in the alternating series test to approximate, within 2 decimal place accuracy, the following values. The exact value of each series is given only as trivia.

   (a) $\sin 1 = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots$

   (b) $\cos(1/2) = 1 - \frac{(1/2)^2}{2!} + \frac{(1/2)^4}{4!} - \frac{(1/2)^6}{6!} + \cdots$

4. Suppose we want to approximate $\ln 2$ with 2 digits of accuracy. If we use the alternating series test,

   (a) How many terms of $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ are needed?

   (b) How many terms of $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}$ are needed?

   Hint. We want the first n for which $|a_{n+1}| < 0.005$. □

Series: Miscellaneous

1. Define $S = 1 + 2/3 + 3/3^2 + 4/3^3 + 5/3^4 + \cdots$.

   (a) Show that the series converges.

   (b) Write out a series for $3S$.

   (c) Substract the given equation from the one you just wrote.

   (d) Evaluate S using the previous part.

   Hint. Use the ratio test. Follow the steps to get $S = 9/4$. □

2. There is a constant γ, called the Euler-Mascheroni constant, so that

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + \epsilon_n,$$

   where $\epsilon_n \to 0$ when $n \to \infty$. Use this fact to answer the following.

   (a) Use the above formula to show that $\sum(1/n)$ diverges.

   (b) Evaluate $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right)$.

   Solution. First, $\sum_0^\infty 1/n = \gamma + \lim_N \ln N = \infty$. Second,

$$\begin{aligned}\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} &= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} \\ &= \ln(2n) + \gamma + \epsilon_{2n} - \ln n - \gamma - \epsilon_n \\ &= \ln 2 + \epsilon_{2n} - \epsilon_n \\ &\to \ln 2\end{aligned}$$

   as $n \to \infty$. □

5. The Fibonacci numbers form a sequence F_n, where $F_1 = F_2 = 1$ and $F_{n+2} = F_n + F_{n+1}$ for all integers n.

   (a) Use telescoping to evaluate $\sum_{n=1}^{\infty} \frac{F_{n-1}}{F_n F_{n+1}}$.

   (b) It turns out that (amazingly)

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

   Does $\sum F_n^{-1}$ converge?

   Solution. Rewrite $F_{n-1} = F_{n+1} - F_n$ and telescope to get a sum of 1. Using limit comparison to the geometric series $\sum(2/(1+\sqrt{5}))^n$ to see that the second series converges. □

3. In 1914, Ramanujan proved that

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4k)!(1103 + 26,390k)}{(k!)^4 396^{4k}}.$$

   This converges by the ratio test.